AN EXTENSION OF THE HARDY–LITTLEWOOD–PÓLYA INEQUALITY

Congming Li, John Villavert

Department of Applied Mathematics
University of Colorado at Boulder, Boulder, CO, USA 80309

Abstract

The Hardy–Littlewood–Pólya (HLP) inequality [1] states that if $a \in l^p$, $b \in l^q$ and

$$p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} > 1,\ \lambda = 2 - \left(\frac{1}{p} + \frac{1}{q}\right),$$

then

$$\sum_{r \neq s} \frac{a_r b_s}{|r-s|^\lambda} \leq C_{p,q} \|a\|_p \|b\|_q.$$

In this article, we prove the HLP inequality in the case where $\lambda = 1, p = q = 2$ with a logarithm correction, as conjectured by Ding [2]:

$$\sum_{r \neq s, 1 \leq r,s \leq N} \frac{a_r b_s}{|r-s|^\lambda} \leq (2 \ln N + 1) \|a\|_2 \|b\|_2.$$

In addition, we derive an accurate estimate for the best constant for this inequality.

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1. Introduction

The well-known Hardy–Littlewood–Sobolev (HLS) inequality states that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\lambda} \, dx \, dy \leq C_{r,\lambda,n} \|f\|_r \|g\|_s \tag{1}$$

Email addresses: congming.li@colorado.edu (Congming Li), john.villavert@colorado.edu (John Villavert)

for any $f \in L^r(\mathbb{R}^n)$ and $g \in L^s(\mathbb{R}^n)$ provided that

$$0 < \lambda < n, 1 < r, s < \infty \text{ with } \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} = 2.$$

Hardy and Littlewood also introduced a double weighted inequality which was later generalized by Stein and Weiss [3]:

$$\Big| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} \, dx \, dy \Big| \leq C_{\alpha,\beta,r,\lambda,n} \|f\|_r \|g\|_s \tag{2}$$

where $1 < r, s < \infty$, $0 < \lambda < n$, $\alpha + \beta \geq 0$,

$$1 - \frac{1}{r} - \frac{\lambda}{n} < \frac{\alpha}{n} < 1 - \frac{1}{r} \text{ and } \frac{1}{r} + \frac{1}{s} + \frac{\lambda + \alpha + \beta}{n} = 2.$$

To obtain the best constant in the weighted Hardy–Littlewood–Sobolev (WHLS) inequality (2), one can maximize the functional

$$J(f,g) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} \, dx \, dy$$

with the constraints $\|f\|_r = \|g\|_s = 1$. On the other hand, the Hardy–Littlewood–Pólya (HLP) inequality[1, inequality 381, p.288] [4]—a discrete analogue of the HLS inequality—is provided in the setting of l^p-spaces. More precisely, the HLP inequality states that if $a \in l^p$, $b \in l^q$ and

$$p > 1, \; q > 1, \; \frac{1}{p} + \frac{1}{q} > 1, \; \lambda = 2 - \left(\frac{1}{p} + \frac{1}{q} \right),$$

then

$$\sum_{r \neq s} \frac{a_r b_s}{|r-s|^\lambda} \leq C \|a\|_p \|b\|_q \tag{3}$$

where the constant C depends on p and q only.
The following theorem was conjectured by X. Ding [2]. It can be regarded as an extension of the well-known HLP inequality in the case $p = q = 2$ and $\lambda = 1$ with a logarithm correction:

Theorem 1. *Let $p = q = 2$ and $\lambda = 2 - \frac{1}{p} - \frac{1}{q} = 1$. If $a, b \in l^p$, then*

$$\sum_{r \neq s, 1 \leq r,s \leq N} \frac{a_r b_s}{|r-s|} \leq 2(\ln N + 1) \|a\|_2 \|b\|_2. \tag{4}$$

In fact we shall prove instead the following theorem in which theorem 1 is a consequence.

Theorem 2. *Let*

$$\lambda_N = \max_{\sum a_r^2 = \sum b_r^2 = 1} \sum_{r \neq s, 1 \leq r,s \leq N} \frac{a_r b_s}{|r-s|}, \tag{5}$$

then

$$2 \ln N - 2 \leq \lambda_N \leq 2 \ln N + 2(1 - \ln 2).$$

Consequently we have:

$$\lambda_N < 2 \ln N + 1.$$

2. Proof of Theorem 2

We prove theorem 2 in three main steps.
In step 1, we choose $a_r = b_r = \frac{1}{\sqrt{N}}$ and calculate that

$$\sum_{r \neq s, 1 \leq r,s \leq N} \frac{a_r b_s}{|r-s|} \geq 2 \ln N - 2$$

This shows that $\lambda_N \geq 2 \ln N - 2$.
In step 2, we derive the Euler-Lagrange equations for the maximizers $\overline{a}$ and $\overline{b}$.
In step 3, we use the Euler-Lagrange equations to show that

$$\lambda_N \leq 2 \ln N + 2(1 - \ln 2),$$

thus completing the proof. The calculations in steps 1 and 3 will make use of the following inequalities. For a positive integer M, we have that

$$\ln(M+1) \leq \sum_{l=1}^{M} \frac{1}{l} \leq 1 + \ln M$$

and

$$\sum_{l=1}^{M} \ln l \geq M \ln M - M + 1.$$

Step 1: Let $a_r = b_r = \frac{1}{\sqrt{N}}$, then $\sum a_r^2 = \sum b_r^2 = 1$ where the summation is from 1 to N. It follows that

$$
\begin{aligned}
\sum_{r\neq s, 1\leq r,s\leq N} \frac{a_r b_s}{|r-s|} &= \frac{1}{N} \sum_{r\neq s, 1\leq r,s\leq N} \frac{1}{|r-s|} \\
&= \frac{2}{N} \sum_{s=1}^{N-1} \sum_{r=s+1}^{N} \frac{1}{r-s} = \frac{2}{N} \sum_{s=1}^{N-1} \sum_{l=1}^{N-s} \frac{1}{l} \\
&\geq \frac{2}{N} \sum_{s=1}^{N-1} \ln(N-s+1) = \frac{2}{N} \sum_{l=1}^{N} \ln l \\
&\geq \frac{2}{N}(N\ln N - N + 1) \\
&\geq 2(\ln N - 1).
\end{aligned}
$$

Using the definition of λ_N along with the preceding calculations, we arrive with the following estimate:

$$\lambda_N \geq 2\ln N - 2. \tag{6}$$

Step 2: We derive the Euler-Lagrange equations for the maximizers of (5). Let

$$J_N(a,b) = \sum_{r\neq s, 1\leq r,s\leq N} \frac{a_r b_s}{|r-s|} - \lambda_N \sqrt{\sum_{1\leq r\leq N} a_r^2 \sum_{1\leq s\leq N} b_s^2}. \tag{7}$$

Then by our definition of λ_N, we have $J_N(a,b) \leq 0$, and by compactness, there exist elements $\overline{a}$ and $\overline{b}$ with $\|\overline{a}\|_2 = \|\overline{b}\|_2 = 1$ such that

$$J_N(\overline{a}, \overline{b}) = 0.$$

Thus, we must have $0 = \frac{d}{da_r} J_N(a,b)\Big|_{(a=\overline{a}, b=\overline{b})}$.
Taking the derivative directly in (7) about $\overline{a}_r$, we obtain:

$$\sum_{s\neq r, 1\leq s\leq N} \frac{\overline{b}_s}{|r-s|} - \lambda_N \overline{a}_r = 0.$$

Similarly, taking the derivative about $\overline{b}_s$, we obtain:

$$\sum_{r\neq s, 1\leq r\leq N} \frac{\overline{a}_r}{|r-s|} - \lambda_N \overline{b}_s = 0.$$

Combining the above two equations together, we obtain the Euler-Lagrange equations:

$$\begin{cases} \lambda_N \overline{a}_r = \displaystyle\sum_{s\neq r, 1\leq s\leq N} \frac{\overline{b}_s}{|r-s|} \\ \lambda_N \overline{b}_s = \displaystyle\sum_{r\neq s, 1\leq r\leq N} \frac{\overline{a}_r}{|r-s|}. \end{cases} \tag{8}$$

Step 3: Here we will show that $\lambda_N \leq 2\ln N + 2(1-\ln 2)$.
With a change of sign if necessary, we may assume that

$$a_{r_0} = \max\{|\overline{a}_r|, |\overline{b}_s| \ : 1 \leq r, s \leq N\} > 0.$$

In fact, we may assume that all components are non-negative (and consequently positive by (8)), and a_{r_0} is the maximum for some r_0. Then

$$\begin{aligned} \lambda_N &= \sum_{s\neq r_0, s=1}^{N} \frac{\overline{b}_s}{a_{r_0}|r_0-s|} \leq \sum_{s\neq r_0, s=1}^{N} \frac{|\overline{b}_s|}{|a_{r_0}||r_0-s|} \leq \sum_{s\neq r_0, s=1}^{N} \frac{1}{|r_0-s|} \\ &= \sum_{s=1}^{r_0-1} \frac{1}{r_0-s} + \sum_{s=r_0+1}^{N} \frac{1}{s-r_0} = \sum_{l=1}^{r_0-1} \frac{1}{l} + \sum_{l=1}^{N-r_0} \frac{1}{l} \\ &\leq 2 + \ln(r_0-1) + \ln(N-r_0) = 2 + \ln((r_0-1)(N-r_0)) \\ &\leq 2 + \ln\left(\frac{N-1}{2}\right)^2 \leq 2 + \ln\left(\frac{N}{2}\right)^2 = 2 + 2(\ln N - \ln 2). \end{aligned}$$

Hence

$$\lambda_N \leq 2\ln N + 2(1-\ln 2). \tag{9}$$

Combining the estimates (6) and (9) yields

$$2\ln N - 2 \leq \lambda_N \leq 2\ln N + 2(1-\ln 2).$$

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